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HARMONIC ANALYSIS ON FUNCTIONS WITH BOUNDED MEANS

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1. Introduction.

The study of the averages $\frac{1}{2T}\int_{-T}^{T}|f|^2$, $\frac{1}{2T}\int_{-T}^{T}f(x+\tau)\bar{f}(x)dx$ began in the 20's. Bohr, Besicovitch, Stepanoff used them to investigate the structure and spectrum of the almost periodic functions $\left(\sum a_n e^{i\alpha_n x},\ \alpha_n \in \mathbf{R}\right)$. Wiener, on the other hand, showed that if μ is a bounded regular Borel measure on $\mathbf{R}$, then μ is continuous if and only if $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\hat{\mu}|^2 = 0$.

The most important contribution to the subject was, yet, due to Wiener in the 30's. Earlier in the twentieth century, some physicists (e.g., Raleigh, Schuster, Taylor) tried to apply harmonic analysis to study the chaotic signals of white light. These types of signal were known to have finite power ($\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f|^2 < \infty$) but infinite energy ($\int_{-\infty}^{\infty}|f|^2 = \infty$). The classical analysis of $L^2(\mathrm{R})$ is hence not suitable for analyzing such signals, nor do the almost periodic functions mentioned above (which have discrete spectra). It was this intellectually deep but mathematically nonrigorous physics problem that led Wiener to his celebrated work of generalized harmonic analysis on functions with bounded means [16]. During the Second World War, Wiener made another stunning discovery by his generalized harmonic analysis: the invention of the theory of prediction and filtering.

The formulation of the quadratic average and the corresponding spectrum in Wiener's work depends on the identity

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(1.1) $$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2\, dx \;=\; \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\int_{-\infty}^{\infty} |g(u+\varepsilon) - g(u-\varepsilon)|^2\, du$$

where $g = W(f)$ is the integrated Fourier transformation (Wiener transformation) of f defined by

(1.2) $$g(u) \;=\; \frac{1}{2\pi}\left(\left(\int_{-\infty}^{-1} + \int_{1}^{\infty} f(x)\, \frac{e^{iux}}{ix}\, dx\right) \;+\; \int_{-1}^{1} f(x)\, \frac{e^{iux}-1}{ix}\, dx\right)$$

In the sense of Schwartz distribution, $(Wf)'$ is the Fourier transformation of f.

The class of functions in (1.1), however, is not closed under addition. Recently Lau and Lee [15] considered the following two larger linear spaces: the Marcinkiewicz space,

$$\mathcal{B}^p \;=\; \{\, f \in L^p_{loc}(\mathbb{R}) : \|f\| = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^p\, dx\right)^{1/p} \;<\; \infty \,\}$$

and the space of functions of bounded p–variation:

$$\mathcal{V}^p \;=\; \{\, g \in L^p_{loc}(\mathbb{R}) : \|g\| = \overline{\lim_{\varepsilon\to 0+}} \left(\frac{1}{2\varepsilon}\int_{-\infty}^{\infty} |g(u+\varepsilon) - g(u-\varepsilon)|^p\, du\right)^{1/p} \;<\; \infty \,\},$$

where $1 < p < \infty$. After proving a Tauberian theorem for limit supremum, they are able to extend (1.1) as

(1.3) $$c_1\|Wf\|_{\mathcal{V}^2} \;\le\; \|f\|_{\mathcal{B}^2} \;\le\; c_2\|Wf\|_{\mathcal{V}^2}, \qquad f \in \mathcal{B}^2,$$

and the best constants c_1, c_2 were found. The convolution operator and the characterization of multipliers on $\mathcal{B}^2$ had also been investigated by Bertrandias [1] and Lau [13, 14].

The spaces $\mathcal{B}^p$ and $\mathcal{V}^p$ involve functions with large equivalent classes (e.g., for $f \in \mathcal{B}^p$, $[f] = \{h : \|f - h\|_{\mathcal{B}^2} = 0\} \supseteq f + L^p$, and for $g \in \mathcal{V}^p$, $[g] = \{h : \|g - h\|_{\mathcal{V}^2} = 0\} \supseteq g + \{h : h \in L^p,$ h' exists$\}$ [15]), hence it is instructive to consider the two more conventional classes of functions

$$B^p \;=\; \{\, f \in L^p_{loc}(\mathbb{R}) : \; \|f\|_{B^p} = \sup_{T\ge 1} \left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^p\, dx\right)^{1/p} \;<\; \infty \,\}$$

and

$$V^p \;=\; \{\, g \in L^p_{loc}(\mathbb{R}) : \; \|g\|_{V^p} \;=\; \sup_{1\ge\varepsilon>0} \left(\frac{1}{2\varepsilon}\int_{-\infty}^{\infty} |g(u+\varepsilon) - g(u-\varepsilon)|^p\, du\right)^{1/p} \;<\; \infty \,\}.$$

The spaces $\mathcal{B}^p$ and $\mathcal{V}^p$ can then be regarded as the quotients of B^p/B^p_0 and V^p/V^p_0, where

$$B^p_0 \;=\; \{\, f \in B^p : \; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^p\, dx \;=\; 0 \,\}$$

and

$$V^p_0 \;=\; \{\, g \in V^p : \; \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\int_{-\infty}^{\infty} |g(u+\varepsilon) - g(u-\varepsilon)|^p\, du \;=\; 0 \,\}.$$

The spaces B^p had been considered by Beurling in [2]. He defined a class of functions A^p,

$1 \le p < \infty$, as

$$A^p = \{ f : \|f\|_{A^p} = \inf_{\omega \in \Omega} \left(\int |f|^p \, \omega^{-(p-1)} \right)^{1/p} < \infty \},$$

where Ω is the set of bounded, positive, integrable even functions ω which are nonincreasing on $\mathbf{R}^+$ and

$$\omega(0) + \int_{-\infty}^{\infty} \omega(x) dx = 1.$$

It is easy to show that $A^1 = L^1$, and A^p can be continuously embedded into L^1 and L^p.

THEOREM 1.1. (Beurling) *For* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, A^p *is a Banach algebra under convolution, and* $(A^p)^*$ *is isomorphic to* B^q.

For $f \in B^p$, the Hilbert transformation Hf, where

$$Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{1}{1+t^2} \right) f(t) \, dt, \quad x \in \mathbf{R},$$

of f is not necessary to be in B^p. A natural space which contains B^p, and closed under the Hilbert transformation is the following:

$$CMO^p = \{ f : \|f\|_{*,p} = \sup_{T \ge 1} \left(\frac{1}{2T} \int_{-T}^{T} |f - m_T f|^p \right)^{1/p} < \infty \},$$

where $m_T f = \frac{1}{2T} \int_{-T}^{T} f$ (*CMO* stands for *Central Mean Oscillation*). This space is analogous to *BMO*, but only takes average on intervals $[-T, T]$. By regarding A^p–B^p as an L^1–L^∞ analog (rather than the L^p–L^q analog), we develop, in section 2 and section 3, the theory of the dual pair H_{A^p}–CMO^q $(1 < p \le 2)$, which is corresponding to the Fefferman–Stein's H^1–*BMO* pair (Theorem 3.4). At the same time, we also obtain a theorem analogous to the Burkholder, Gundy and Silverstein theorem (Theorem 3.5).

While the supremum part of the norm of B^p dominates for duality and Hilbert transformation, the p–module of the norm dominates the Fourier transformation. Indeed in section 4, we will give a result similar to (1.3) as an analog of the Plancherel theorem for the Wiener transformation on B^2 (Theorem 4.3).

The details of proofs will appear elsewhere ([7], [8]).

2. Harmonic extensions and CMO^p.

Unless otherwise specified, we assume $1 < p < \infty$.

Let $P_y(x) = \frac{1}{\pi(x^2+y^2)}$ be the Poisson kernel, and let $u(z) = u_y(x) = P_y*f(x)$, where $z = x + iy$, be the harmonic extension of f on the upper half plane $\mathbf{R}^2_+$. Follwoing a standard procedure, we can prove

PROPOSITION 2.1. *Let $f \in B^p$ or A^p, then*

(i) u_y converges to f nontangentially a.e. (and also converges in norm for the A^p case);

(ii) there exists $c > 0$ such that $\|u_y\| \le c\|f\|$, $\forall\, y > 0$.

PROPOSITION 2.2. *Let $u(z) = u_y(x)$ be a harmonic function on $\mathbf{R}^2_+$, then*

$$\sup_{y>0} \|u_y\|_{B^p} < \infty \quad (or \sup_{y>0} \|u_y\|_{A^p} < \infty)$$

if and only if there exists $f \in B^p$ (or A^p) such that $u(z) = P_y*f(x)$.

We define H_{B^p} to be the class of analytic functions $u(z)$ on $\mathbf{R}^2_+$ such that

$$\|u\|_{H_{B^p}} = \sup_{y>0} \|u_y\|_{B^p} < \infty.$$

Similarly we can define H_{A^p}.

Let Mf be the *Hardy–Littlewood maximal function* of f defined by

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|} \int_I |f|,$$

where I is an interval containing x.

THEOREM 2.3. *For any $f \in B^p$, there exists $c > 0$, such that*

$$\|Mf\|_{B^p} \le c\|f\|_{B^p}.$$

However such an inequality does not hold for $f \in A^q$ (e.g. take $f = \chi_{[-1,\,1]}$). For this we consider the *nontangential maximal function*

$$f^*(x) = \sup_{(t,y)\in\Gamma(x)} |u(t,y)|,$$

where $\Gamma(x) = \{\ z = x+iy : |x-t| < y\ \}$. By using a duality inequality of Fefferman and Stein [10] (for $r > 1$, there exists c such that for any f, $\phi \ge 0$, $\int (Mf)^r \phi \le c\int |f|^r M\phi$), Theorem 2.3, and a factorization theorem of analytic functions, we prove

THEOREM 2.4. *For $f \in H_{A^p}$, $\|f^*\|_{A^p} \le c\,\|f\|_{H_{A^p}}$.*

Let CMO^p be defined as in section 1. By identifying constant functions, it is easy to show that $B^p \subsetneq CMO^p$, and the inclusion is proper (e.g. $f(x) = \ln|x|$, then $f \in CMO^p \setminus B^p$). If f is an odd function, then $f \in CMO^p$ implies that $f \in B^p$ (since $m_T f = 0$ for all $T \geq 1$).

PROPOSITION 2.5. *Let $1 < p_1 < p_2 < \infty$, then $CMO^{p_2} \subsetneq CMO^{p_1}$, and CMO^{p_2} is not dense in CMO^{p_1}.*

The space B^p is not closed under the Hilbert transformation (let $f = \chi_{[0,\infty)}$, then $Hf(x) = \frac{1}{\pi}\ln|x| \notin B^p$). However for CMO^p, we have

THEOREM 2.6. *If $f \in CMO^p$, then $Hf \in CMO^p$ and $\|Hf\|_{*,p} \leq c\|f\|_{*,p}$.*

For $p = 2$, we can characterize CMO^p by an analog of the Carlson measure [12]. A regular Borel measure λ on $\mathbf{R}^2_+$ is called a *Central Carleson Measure* (C.C. measure) if

$$\sup_{T\geq 1} \frac{1}{2T}(\lambda[-T,T]^2) < \infty,$$

where $[-T,T]^2 = [-T,T] \times [-T,T]$.

THEOREM 2.7. *$f \in CMO^2$ if and only if $y|\nabla u(x,y)|^2 dxdy$ is a C.C. measure.*

3. Atomic decomposition and duality.

In this section, our main concern is the duality of H_{A^p} and CMO^q. Following the notion of Coifman and Weiss [9], we call a real integrable function ϕ on R an (a,p)–atom, $1 < p < \infty$, if there exists a bounded interval I centered at 0, with $|I| \geq 2$ such that (*i*) supp $\phi \subseteq I$, (*ii*) $\|f\|_{L^p} \leq |I|^{-(1/q)}$, and (*iii*) $\int_I \phi(x)dx = 0$.

We will use $H^{a,p}$ to denote the class

$$\{ f : f \text{ real}, f = \sum \lambda_k \phi_k, \ \{\phi_k\} \text{ are } (a,p)\text{–atoms}, \sum |\lambda_k| \leq \infty \},$$

and let

$$\|f\|_{a,p} = \inf \{ \sum |\lambda_i| : f = \sum \lambda_i \phi_i \text{ as above } \}.$$

Under this norm, $H^{a,p}$ is a Banach space. A duality argument yields

THEOREM 3.1. *The dual of $H^{a,p}$ is isomorphic to $CMO^q_{\mathbf{R}}$, the subspace of real valued functions in CMO^q.*

The remaining task is to identify $H^{a,p}$ with H_{A^p}. Let $H_{A^p_{\mathbf{R}}}$ denote the class of real valued functions f such that both f and $\tilde{f}$ are in A^p, and

$$\|f\|_{H_{A^p_{\mathbf{R}}}} = \|f\|_{A^p} + \|\tilde{f}\|_{A^p}.$$

It follows from the Open Mapping Theorem that $H_{A^p_{\mathbf{R}}}$ is isomorphic to $H_{A}p$. Let C_p denote the class of real valued functions on R such that both f and f^* are in A^p.

LEMMA 3.2. *For* $1 < p < \infty$, $H^{a,p} \subseteq H_{A^p_{\mathbf{R}}} \subseteq C_p$.

The first inclusion follows by showing that (a,p)–atoms are in $H_{A^p_{\mathbf{R}}}$, so ar their sums. The second inclusion follows from Theorem 2.4.

The more difficult part is the following

LEMMA 3.3. *For* $1 < p \le 2$, $C_p \subseteq H^{a,p}$.

The proof depends on revising a method of Calderón [6] and Wilson [17] to decompose functions in C_p.

By using Theorem 3.1 and the above two lemms, we have

THEOREM 3.4. *For* $1 < p \le 2$, $(H_{A^p})^*$ *is isomorphic to* $CMO^q_{\mathbf{R}}$.

THEOREM 3.5. *For* $1 < p \le 2$, *and for any real* f, $f + i\tilde{f} \in H_{A^p}$ *if and only if* $f^* \in A^p$.

Note that for $p = 1$ $(A^1 = L^1)$, Theorem 3.4 is the Fefferman–Stein duality theorem, and Theorem 3.5 is the Burkholder, Gundy and Silverstein theorem.

We do not know Theorem 3.4 and 3.5 for the case $2 < p < \infty$. As simple corollaries, we have (see [12] for the analogs)

COROLLARY 3.6. *For* $2 \le p < \infty$, $f \in CMO^p$ *if and only if* $f = \psi_1 + H\psi_2 + \alpha$ *where* $\psi_1, \psi_2 \in B^p$, α *is a constant, and*

$$\|\psi_1\|_{B^p}, \|\psi_2\|_{B^p} \le c\|f\|_{*,p}$$

for some constant c.

COROLLARY 3.7. *For* $2 \le p < \infty$, $B^p/H_{B^p} \approx CMO^p_{\mathbf{R}}$.

COROLLARY 3.8. *For* $2 \le p < \infty$, *and for* $f \in B^p$,

$$c_2\|f - iHf\|_{*,p} \le dis(f, H_{B^p}) \le c_1\|f - iHf\|_{*,p},$$

where c_1, c_2 *are absolute constants.*

COROLLARY 3.9. *For* $2 \le p < \infty$, *and for any real* $f \in B^p$, *there exist absolute constants* c_1, c_2

such that

$$c_2 dis(f, H_{B^p_{\mathbf{R}}}) \leq dist_*(Hf, B^p) \leq c_1 dis(f, H_{B^p_{\mathbf{R}}}),$$

where

$$dis(f, H_{B^p_{\mathbf{R}}}) = inf \{ \|f - Re\, g\|_{B^p} : g \in H_{B^p} \}$$

and

$$dist_*(Hf, B^p) = inf \{ \|Hf - g\|_{*,p} : g \in B^p \}.$$

4. The Wiener transformation.

For $1 < p < \infty$, $\alpha > 0$, let

$$B^p_\alpha = \{ f : \|f\| = \sup_{T \geq 1} \left(\frac{1}{(2T)^\alpha} \int_{-T}^{T} |f|^p \right)^{1/p} < \infty \}.$$

Note that for $\alpha = 1$, $B^p_\alpha = B^p$. The above expression is a natural extension of the standard average and had been considered by Bochner [3. Chapter 6] and Wiener [16, p. 161]. For any Borel measurable function f on $\mathbf{R}$, we define the α–th difference $\Delta^\alpha_h f$ by

$$\Delta^\alpha_h f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x - kh)$$

(see e.g. [5]). It is easy to check that for $\alpha, \beta \in \mathbf{R}$,

$$\Delta^\alpha_h \Delta^\beta_h f = \Delta^{\alpha+\beta}_h f,$$

$$\Delta^\alpha_h e^{i(\bullet)x} = (1 - e^{-ihx})^\alpha e^{i(\bullet)x}.$$

For $\alpha > 0$, if we let

$$D^\alpha e^{i(\bullet)x} = \lim_{h \to 0} \frac{\Delta^\alpha_h e^{i(\bullet)x}}{h^\alpha},$$

then $D^\alpha e^{i(\bullet)x} = (ix)^\alpha e^{i(\bullet)x}$. The α–*integral* of $e^{i(\bullet)x}$ is defined as

$$\left(\int_0^u \right)^\alpha e^{ivx} dv = (ix)^{(\alpha)} \left(\int_0^u \right)^{[\alpha]} e^{ivx} dv,$$

where $[\alpha]$ is the smallest integer greater than or equal to α, $(\alpha) = [\alpha] - \alpha$, and $\left(\int_0^u \right)^{[\alpha]} e^{ivx} dv$ is the $[\alpha]$–th integral of e^{iux} in the ordinary sense.

For $f \in B^2_\alpha$, we define $W_\alpha(f)$

$$g(u) = W_\alpha f(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_1^\infty f(x) \frac{e^{iux}}{(ix)^\alpha} dx + \int_{-1}^{1} f(x) \left(\left(\int_0^u \right)^\alpha e^{ivx} dv \right) dx \right).$$

This definition extends (1.2) where $\alpha = 1$, and the one considered by Bochner in [5, Chapter 6] where α is a positive integer.

For $p > 1$, $\alpha > 0$, we let

$$V_\alpha^p = \{ g : \|g\| = \sup_{0<\varepsilon\leq 1} \left(\frac{1}{\varepsilon^\alpha} \int_{-\infty}^{\infty} |\Delta_\varepsilon^\alpha g|^p \right)^{1/p} < \infty \}.$$

THEOREM 4.1. *The map* $W_\alpha : B_\alpha^2 \to V_\alpha^2$ *is a surjective isomorphism and*

$$\|W_\alpha\| = \left(\tilde{h}(0) + \alpha \int_1^\infty x^{\alpha-1} \tilde{h}(x) dx\right)^{1/2}, \quad \|W_\alpha^{-1}\| = (\, h(1)\,)^{-1/2},$$

where $h(x) = c\left|\frac{\sin x}{x}\right|^{2\alpha}$, $\tilde{h}$ *is the smallest decreasing majorant of* h, *i.e.*, $\tilde{h}(x) = \sup_{t\geq x} h(t)$, *and* $c = \dfrac{2^{2\alpha-1}}{\pi}$.

The proof follows from:

(*a*) Identifying $\Delta_\varepsilon^\alpha g(u)$ with

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1-e^{iux}}{ix}\right)^\alpha f(x)\, e^{iux} dx,$$

the Plancherel Theorem will imply

$$\int_{-\infty}^{\infty} |\Delta_h^\alpha g|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{2\sin(hx/2)}{x}\right|^{2\alpha} |f(x)|^2 dx.$$

(*b*) An application of the following extended form of Tauberian theorem:

THEOREM 4.2. *Let* $\alpha > 0$ *and let* h *be a nonnegative continuous function on* $[0, \infty)$.

(*i*) *Suppose* $c_1 = \tilde{h}(0) + \alpha \int_1^\infty x^{\alpha-1} \tilde{h}(x) dx < \infty$, *then for any* $f \geq 0$,

$$\sup_{T\geq 1} \int_0^\infty \frac{f(Tx)}{T^{\alpha-1}} h(x)\, dx \;\leq\; c_1 \sup_{T\geq 1} \frac{1}{T^\alpha} \int_0^T f(x)\, dx.$$

(*ii*) *Let* $c_2 = h(1)$. *Suppose that for all* $x \in [0, 1]$, $h(x) \geq c_2$, *then for* $f \geq 0$,

$$c_2 \sup_{T\geq 1} \frac{1}{T^\alpha} \int_0^T f(x)\, dx \;\leq\; \sup_{T\geq 1} \int_0^\infty \frac{f(Tx)}{T^{\alpha-1}} h(x)\, dx.$$

Moreover, the constants c_1, c_2 *are best estimates for the inequalities.*

Note that $h(x) = \left|\frac{\sin x}{x}\right|^p$, $x \geq 0$, $p > 0$, satisfies the hypothesis in Theorem 4.2.

For $\alpha = 1$, W_α is the Wiener transformation W in (1.2). If we let

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx, \qquad \check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) e^{iux} du$$

be the Fourier transformation of f, and the inverse Fourier transformation of g respectively, then it is simple to show that $(Wf)' = \check{f}$ in the distributional sense. Theorem 4.1 reduces to

THEOREM 4.3. *The Wiener transformation* $W : B^2 \to V^2$ *is a surjective isomorphism with*

$$\|W\| = \left(\tilde{h}(0) + \alpha\int_1^\infty \tilde{h}(x)dx \right)^{1/2},$$

where $h(x) = \frac{2}{\pi}\left|\frac{\sin x}{x}\right|^2$.

We observe that for $f \in L^2$, $g \in L^2$ such that g' is also in L^2,

$$\int_0^\infty \frac{1}{h^2}\left(\int_{-\infty}^\infty \Delta_h^+ f(u)\, \Delta_h^- g(u)\, du\right)dh \qquad (\Delta_h^+ f = f(x+h) - f(x),\ \Delta_h^- g = g(x) - g(x-h))$$

$$= -\frac{1}{2\pi}\int_0^\infty \frac{1}{h^2}\left(\int_{-\infty}^\infty (e^{ihx} - 1)^2 \hat{f}(x)\hat{g}(-x)dx\right)dh$$

$$= -\frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(x)\hat{g}(-x)\left(\int_0^\infty \frac{(e^{ihx}-1)^2}{h^2}\, dh\right) dx$$

$$= -\frac{2\ \ln\ 2}{2\pi}\int_{-\infty}^\infty (-ix)\hat{f}(x)\hat{g}(-x)dx$$

$$= -2\ \ln 2\int_{-\infty}^\infty f(x)dg(x)$$

In view of such an identity, we define for $g \in V^2$,

$$\int_{-\infty}^\infty fdg = c\int_0^\infty \frac{1}{h^2}\left(\int_{-\infty}^\infty \Delta_h^+ f(u)\, \Delta_h^- g(u)\, du\right) dh$$

whenever the integral converges, and $c = (-2 \ln 2)^{-1}$. By using this, we can show that

THEOREM 4.4. *Let* $g \in V^2$, *then*

$$\int_{-\infty}^\infty e^{-iu(\bullet)}dg(u) = W^{-1}(g) \quad a.e.$$

Let A^2 be the convolution algebra as in previous sections, Beurling [2] proved that the Fourier transformation is an isomorphism of A^2 onto

$$U^2 = \{\, l : \|l\| = \|l\|_{L^2} + \int_0^\infty \frac{1}{h^{3/2}}\|\Delta_h l(u)\|_{L^2}\, dh < \infty \,\}.$$

If we define the duality of V^2 and U^2 by

$$\langle g, l\rangle = \int_{-\infty}^\infty \bar{l}\, dg, \qquad g \in V^2, l \in U^2,$$

Theorem 1.1, Theorem 4.3, and Theorem 4.4 imply

THEOREM 4.5. *$(U^2)^*$ is isomorphic to* V^2, *and*

$$\langle g, l \rangle = \langle W^{-1}g, \check{l} \rangle, \qquad g \in V^2, l \in U^2.$$

In terms of a diagram, the duality and the Wiener transformation for B^2 can be represented as:

$$(A^2)^* \approx B^2 \;\underset{W^{-1}}{\overset{W}{\rightleftarrows}}\; V^2 \approx (U^2)^*$$

$$A^2 \;\underset{\vee}{\overset{\wedge}{\rightleftarrows}}\; U^2$$

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